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Operators in a Relativistic Fock Space for Fermions†

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Abstract

We extend the definition of creation and annihilation operators for a particle in an arbitrary state in Fock space to the relativistic case. We pay special attention to the metric in this space and to the phase convention that leads to anticommutation relations for these operators. The same conventions also apply in the case of several types of fermions.

1. *Introduction*

We have seen (Marx, 1970a) how symmetric and antisymmetric components of a vector in Fock space lead to commutation and anticommutation relations respectively for creation and annihilation operators. These operators were defined in terms of normalized functions within a Hilbert space, although we pointed out that much of the usual approach is recovered when we formally substitute Dirac δ-functions in their place.

It is well known that the 'second quantization' formalism is equivalent to quantum mechanics for systems of several particles in the nonrelativistic case, as seen when a Fock space (Fock, 1932) is used. On the other hand, it is generally agreed that quantized fields are necessary in the relativistic case to allow for processes in which particles are created or annihilated. We have shown (Marx, 1969, 1970b, c, d) that this is not so; pair creation and annihilation can be taken into account in a theory of relativistic quantum mechanics with a fixed number of 'particles',§ which can be found propagating forward in time as particles or backward in time as antiparticles. These studies suggest that it is convenient to extend the nonrelativistic formulation in Fock space to the relativistic case in a particular way, which we present below.

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§ We use the quotation marks on a term like 'particles' to indicate that it refers collectively to particles and antiparticles.

The relativistic quantum mechanics of several charged bosons in an external electromagnetic field can be formulated (Marx, 1970c) in terms of a symmetric wave function $\phi^{(n)}(x_1, x_2, \ldots, x_n)$ that obeys a set of second-order differential equations. This implies that the state of the system is not given by the wave function alone, but that a number of time derivatives also have to be specified. An equivalent but more convenient description of the state of the system relies on the set of probability amplitudes $g^{(\kappa_1 \kappa_2 \cdots \kappa_n)}(x_1, x_2, \ldots, x_n)$, which are generalizations of the positive and negative frequency parts† of $\phi^{(n)}$, and the corresponding creation and annihilation operators are also given there.

A similar theory for spin-$\frac{1}{2}$ fermions presents some serious difficulties, due to the fact that the conserved density for the Dirac equation is positive, while our probabilistic interpretation is based on conservation of charge. We have proposed a solution through a modification of the Dirac equation (Marx, 1970c) and another based on the Klein–Gordon equation for two-component spinors (Marx, 1970b). In either case we can use the Fock space briefly presented in the latter.

In the following section we give arguments that lead to our choice of components of the Fock space for relativistic particles, the scalar product in this space and the creation and annihilation operators. In Section 3 we extend these ideas to several different types of particles, and we conclude with some remarks in Section 4. Special attention is given to the phase convention that leads to anticommutation relations for those operators.

The notation we use is explained below or in some of our previous papers (Marx, 1969, 1970a, b, c, d).

2. *Creation and Annihilation Operators for Relativistic Fermions*

We restrict our discussion to spin-$\frac{1}{2}$ fermions in order to be specific, and we deal only with the state vectors that describe the systems, which have no time dependence. The dynamics we have in mind can be based on either form of relativistic quantum mechanics (Marx, 1970b, d).

The components of a vector Ψ in Fock space are probability amplitudes either in configuration or in momentum space. We adopt the notation $\psi^{(\kappa_1 \kappa_2 \cdots \kappa_n)}_{(1, 2, \ldots, n)}$, where a subindex i stands either for the position variable $\mathbf{x}_i$ and the two-valued spinor index A_i, or for the momentum variable $\mathbf{k}_i$ and the helicity index λ_i. These two types of amplitudes are related by

$$g^{(\kappa_1 \cdots \kappa_n)}_{A_1 \ldots A_n}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = (2\pi)^{-3n/2} \int d^3 k_1 \ldots d^3 k_n \, b^{(\kappa_1 \cdots \kappa_n)}_{\lambda_1 \ldots \lambda_n}(\mathbf{k}_1, \ldots, \mathbf{k}_n) \times \chi^{\lambda_1}_{A_1}(\mathbf{k}_1) \ldots \chi^{\lambda_n}_{A_n}(\mathbf{k}_n) \exp\left[i \sum_{j=1}^{n} \mathbf{k}_j . \mathbf{x}_j\right] \qquad (2.1)$$

† The indices κ_i range over + and −, and these amplitudes obey symmetry relations derived from the symmetry of $\phi^{(n)}$.

where the χ are the two-component helicity states. These amplitudes are antisymmetric under the exchange of the sets of variables that correspond to the same mode of propagation, and the exchange of the variables that correspond to different modes gives a related amplitude with a factor of -1. The amplitude in momentum space is the one generally used, possibly with the helicity states replaced by spin up and spin down, but there is no agreement about the amplitude in configuration space. To a certain extent, it is a matter of definition, but the amplitudes defined in equation (2.1) lead to a probability density with the usual properties (Schröder, 1964; Marx, 1968, 1970b). This is not the case with the expressions used by Schweber (1961).

The precise definitions of the creation and annihilation operators depend on the choice of a scalar product in Fock space, which in turn is suggested by the dynamical theory to be used. We choose

$$(\Psi, \Psi') = \psi^{(0)*}\psi'^{(0)} + (\psi^{(\kappa_1)}, \psi'^{(\kappa_1)}) + (\psi^{(\kappa_1 \kappa_2)}, \psi'^{(\kappa_1 \kappa_2)}) + \dots \qquad (2.2)$$

where we have to sum over the repeated indices κ_i, and the scalar products of the amplitudes on the right-hand side implies summations over spin indices and integrations over continuous variables. We explicitly consider all possible values of the κ_i, not just those amplitudes in which the particle variables come before the antiparticle variables, although the latter suffice to specify the state. We have two reasons to do this: It simplifies the writing of the equations of motion for the amplitudes, and it makes the connection between conservation of charge and the norm of the state† possible.

We give the creation and annihilation operators for a particle or an antiparticle in a state b given by a normalized two-component vector in a Hilbert space. We set

$$\Psi'_{\pm} = R^{(\pm)}(b)\Psi \qquad (2.3)$$

$$\Psi''_{\pm} = L^{(\pm)}(b)\Psi \qquad (2.4)$$

and we have to specify only the components with p particle variables followed by m antiparticle variables. We have‡

$$\psi'_{+\,(1,\dots,p,p+1,\dots,n)}{}^{(+\dots+-\dots-)} = n^{-1/2}[(p-1)!]^{-1}\,\epsilon^{i_1\, i_2 \dots i_p}_{1,\, 2, \dots, p}\, b_{i_1}\, \psi_{(i_2 \dots i_p, p+1, \dots n)}{}^{(+\dots+-\dots-)} \qquad (2.9)$$

† When the given 'boundary' conditions specify p particles at the initial time and m antiparticles at the final time, there are $C_p{}^n$ ways to choose the particle variables out of the total number n of variables. Thus, the norm of this amplitude is $p!\,m!(n!)^{-1}$ and the norm of the vector in Fock space is 1. The corresponding theory for scalar bosons was presented before (Marx, 1970c).

‡ In the equations that follow we use the summation convention in a flexible way. The indices of the Levi–Civita tensor range over the integers from 1 to p or from $p+1$ to n, but when the index is attached to a function, it represents the spin and continuous variables. If the repeated index stands on two functions, a summation over spin indices and an integration over continuous variables is implied. Thus, for instance

$$\psi'_{+\,\lambda_1\lambda_2}^{(++)}(\mathbf{k}_1, \mathbf{k}_2) = 2^{-1/2}[b_{\lambda_1}(\mathbf{k}_1)\,\psi^{(+)}_{\lambda_2}(\mathbf{k}_2) - b_{\lambda_2}(\mathbf{k}_2)\,\psi^{(+)}_{\lambda_1}(\mathbf{k}_1)] \qquad (2.5)$$

$$\begin{aligned}\psi'_{-\,\lambda_1\lambda_2\lambda_3\lambda_4}^{(+---)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = -4^{-1/2}[&b_{\lambda_2}(\mathbf{k}_2)\,\psi^{(+--)}_{\lambda_1\lambda_3\lambda_4}(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4) - b_{\lambda_3}(\mathbf{k}_3)\,\psi^{(+--)}_{\lambda_1\lambda_2\lambda_4}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4) \\ &+ b_{\lambda_4}(\mathbf{k}_4)\,\psi^{(+--)}_{\lambda_1\lambda_2\lambda_3}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\end{aligned} \qquad (2.6)$$

$$\psi'_{-\,(1,\ldots,p,p+1,\ldots,n)}^{\;(+\ldots+-\ldots-)} = (-1)^p n^{-1/2}[(m-1)!]^{-1}\, \epsilon_{p+1,\,p+2,\ldots,n}^{\;i_1\,i_2\ldots i_m}\, b_{i_1} \times \psi_{(1,\ldots,p,i_2\ldots i_m)}^{(+\ldots+-\ldots-)} \tag{2.10}$$

$$\psi''_{+\,(1,\ldots,p,p+1,\ldots,n)}^{\;(+\ldots+-\ldots-)} = (n+1)^{1/2}\, b_i^*\, \psi_{(i,1,\ldots,p,p+1,\ldots,n)}^{(++\ldots+-\ldots-)} \tag{2.11}$$

$$\psi''_{-\,(1,\ldots,p,p+1,\ldots,n)}^{\;(+\ldots+-\ldots-)} = (-1)^p (n+1)^{1/2}\, b_i^*\, \psi_{(1,\ldots,p,i,p+1,\ldots,n)}^{(+\ldots+--\ldots-)} \tag{2.12}$$

Annihilation operators are the Hermitian conjugates of creation operators with respect to the metric (2.2), and they satisfy the anticommutation relations

$$\{R^{(\kappa)}(b), R^{(\kappa')}(b')\} = 0 \tag{2.13}$$

$$\{L^{(\kappa)}(b), L^{(\kappa')}(b')\} = 0 \tag{2.14}$$

$$\{L^{(\kappa)}(b), R^{(\kappa')}(b')\} = \delta^{\kappa\kappa'}(b, b') \tag{2.15}$$

We note that the states Ψ' and Ψ'' are not necessarily normalized for an arbitrary b, and that it is the phase factor $(-1)^p$ in equations (2.10) and (2.12) that makes the operators for particles and antiparticles anticommute; without it they would commute.

If we set

$$b_{\lambda''}(\mathbf{k}'') = \delta_{\lambda\lambda''}\, \delta(\mathbf{k} - \mathbf{k}'') \tag{2.16}$$

we obtain operators $R_\lambda^{(\kappa)}(\mathbf{k})$ and $L_\lambda^{(\kappa)}(\mathbf{k})$, whose nonvanishing anticommutators are given by

$$\{L_\lambda^{(\kappa)}(\mathbf{k}), R_{\lambda'}^{(\kappa')}(\mathbf{k}')\} = \delta^{\kappa\kappa'}\, \delta_{\lambda\lambda'}\, \delta(\mathbf{k} - \mathbf{k}') \tag{2.17}$$

with similar results in configuration space, where we use δ-functions as in the nonrelativistic case, not the $S^{(+)}$ used by Schweber (1961).

3. *Several Types of Relativistic Fermions*

It is customary to assume that creation and annihilation operators corresponding to different types of particles also anticommute. This is no longer directly related to the statistics obeyed by the particles, and the field operators are not observables in quantum theory anyway.

In the case of different particles, no useful purpose is served by mixing the sets of variables. The appropriate phase factor that makes the operators anticommute is easily determined after the types of particles are numbered.

$$\psi''_{+\,\lambda_1\lambda_2}^{\;(++)}(\mathbf{k}_1, \mathbf{k}_2) = 3^{1/2}\, b_\lambda^*(\mathbf{k})\, \psi_{\lambda\lambda_1\lambda_2}^{(+++)}(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2) \tag{2.7}$$

$$\psi''_{-\,\lambda_1\lambda_2\lambda_3\lambda_4}^{\;(+---)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = -5^{1/2}\, b_\lambda^*(\mathbf{k})\, \psi_{\lambda_1\lambda\lambda_2\lambda_3\lambda_4}^{(+----)}(\mathbf{k}_1, \mathbf{k}, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \tag{2.8}$$

The different scalar product used by Schweber leads to different normalization factors, which in our notation would be $p^{-1/2}$, $m^{-1/2}$, $(p+1)^{1/2}$ and $(m+1)^{1/2}$ in equations (2.9)–(2.12). Since the dynamics we have in mind selects a subspace with a fixed number of variables, we could introduce factors of $(n!)^{-1}$ in equation (2.2), as in Marx (1970a), which eliminates the factors $n^{-1/2}$ and $(n+1)^{1/2}$ from these equations.

We can use equations (2.9)–(2.12) for a type r particle if we ignore the effect of the other particles, except for the phase factor $(-1)^{n_1+n_2+\cdots+n_{r-1}}$.

It is not obvious, though, precisely which are different particles. Protons and neutrons are certainly different when electromagnetic interactions are considered, but they appear as two different states of the nucleon, distinguished by the values of an isospin index, as far as strong interactions are concerned. Even more particles can be included if we deal with unitary spin. These alternatives do make a difference in what we have called normalization factors. The component of the vector with n_1 'protons' n_2 'neutrons' in the equation for the creation operator for one proton has a factor $(n_1)^{-1/2}$ if we consider them as different types of particles, and $(n_1+n_2)^{-1/2}$ if they are states of the same particle. In the latter case any of the n_1+n_2 variables can correspond to a proton or a neutron. This is naturally also reflected in the value of the norm of the vector, that is, the number of times a term is repeated in the scalar product.

From the discussion above we conclude that the precise form of the Fock space and the operators depends on the nature of the dynamical problem we are considering.

4. *Concluding Remarks*

We have presented our definitions of creation and annihilation fermion operators in a relativistic Fock space based on probability amplitudes in momentum or configuration space. Although we did not consider the dynamical development of the state vector, we determined the precise form of the scalar product and operators in terms of a relativistic quantum mechanics with a fixed number of 'particles', which is closer to the non-relativistic formulation than the usual relativistic quantum field theories. It can be based on conservation of charge, of baryon number or even angular momentum, which makes fermion lines in Feynman diagrams continuous. Such a theory would exclude closed fermion loops, which are troublesome in the perturbation expansion of cross sections; it remains to be seen whether a complete dynamical theory that agrees with experiment can be formulated under these terms.

We have restricted our discussion to spin-$\frac{1}{2}$ fields, but we foresee no difficulties in extending this formalism to particles with higher spin, if necessary.

Boson fields are associated to particles with integer spin, that can be created and annihilated singly. Such a quantized field would correspond more closely to the usual approach, in which the number of particles is not fixed.

We have also emphasized the phase factors that lead to anticommutation relations for operators of essentially independent particles; such a discussion is much simpler in Fock space than in occupation-number space.

These considerations are preliminary to the solution of much more difficult problems, such as a mathematically sound theory of quantum electrodynamics or a simpler formulation of bound state problems in relativistic quantum mechanics or quantum field theory.

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